Evolution of Second-Order Cosmological Perturbations and Non-Gaussianity

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We present a second-order gauge-invariant formalism to study the evolution of curvature perturbations in a Friedmann-Robertson-Walker universe filled by multiple interacting fluids. We apply such a general formalism to describe the evolution of the second-order curvature perturbations in the standard one-single field inflation, in the curvaton and in the inhomogeneous reheating scenarios for the generation of the cosmological perturbations. Moreover, we provide the exact expression for the second-order temperature anisotropies on large scales, including second-order gravitational effects and extend the well-known formula for the Sachs-Wolfe effect at linear order. Our findings clarify what is the exact non-linearity parameter $f_{\rm NL}$ entering in the determination of higher-order statistics such as the bispectrum of Cosmic Microwave Background temperature anisotropies. Finally, we compute the level of non-Gaussianity in each scenario for the creation of cosmological perturbations.

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I. INTRODUCTION

Inflation [1,2] has become the dominant paradigm for understanding the initial conditions for structure formation and for Cosmic Microwave Background (CMB) anisotropy. In the inflationary picture, primordial density and gravity-wave fluctuations are created from quantum fluctuations "redshifted" out of the horizon during an early period of superluminal expansion of the universe, where they are "frozen" [3–7]. Perturbations at the surface of last scattering are observable as temperature anisotropy in the CMB, which was first detected by the COsmic Background Explorer (COBE) satellite [8–10]. The last and most impressive confirmation of the inflationary paradigm has been recently provided by the data of the Wilkinson Microwave Anistropy Probe (*WMAP*) mission which has marked the beginning of the precision era of the CMB measurements in space [11]. The *WMAP* collaboration has produced a full-sky map of the angular variations of the CMB, with unprecedented accuracy. *WMAP* data confirm the inflationary mechanism as responsible for the generation of curvature (adiabatic) super-horizon fluctuations.

Despite the simplicity of the inflationary paradigm, the mechanism by which cosmological adiabatic perturbations are generated is not established. In the *standard scenario* associated to one-single field models of inflation, the observed density perturbations are due to fluctuations of the inflaton field itself. When inflation ends, the inflaton oscillates about the minimum of its potential and decays, thereby reheating the universe. As a result of the fluctuations each region of the universe goes through the same history but at slightly different times. The final temperature anisotropies are caused by the fact that inflation lasts different amounts of time in different regions of the universe leading to adiabatic perturbations. Under this hypothesis, the *WMAP* dataset already allows to extract the parameters relevant for distinguishing among single-field inflation models [12].

An alternative to the standard scenario is represented by the *curvaton mechanism* [13–16] where the final curvature perturbations are produced from an initial isocurvature perturbation associated to the quantum fluctuations of a light scalar field (other than the inflaton), the curvaton, whose energy density is negligible during inflation. The curvaton isocurvature perturbations are transformed into adiabatic ones when the curvaton decays into radiation much after the end of inflation. Recently, another mechanism for the generation of cosmological perturbations has been proposed [17–19] dubbed the *inhomogeneous reheating scenario* (sometimes called the modulated reheating scenario)*. It acts during the reheating stage after inflation if super-horizon spatial fluctuations in the decay rate of the inflaton field are induced during inflation, causing adiabatic perturbations in the final reheating temperature in different regions of the universe.

Contrary to the standard picture, the curvaton and the inhomogeneous reheating scenario mechanism exploit the fact that the total curvature perturbation (on uniform density hypersurfaces) ζ can change on arbitrarily large scales due to a non-adiabatic pressure perturbation which may be present in a multi-fluid system [21–25]. While the entropy perturbations evolve independently of the curvature perturbation on large scales, the evolution of the large-scale curvature is sourced by entropy perturbations.

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*The idea that the total curvature perturbation may be affected on large scales by entropy perturbations when there exists a scalar field affecting particle masses or couplings constants controlling the reheating process has first been suggested in [20].

Fortunately, the standard and the curvaton scenarios have different observational signatures. The curvaton and the inhomogeneous reheating scenario allow to generate the observed level of density perturbations with a much lower scale of inflation and thus generically predict a smaller level of gravitational waves.

More interestingly, the various scenarios for the generation of the cosmological perturbations predict different levels of non-Gaussianity which is usually parametrized by a dimensionless non-linear parameter $f_{\rm NL}$. The non-Gaussianity can be large enough to be detectable either by present CMB experiments, the current *WMAP* [26] bound on non-Gaussianity being $|f_{\rm NL}| \lesssim 10^2$, or by future planned satellites,such as *Planck*, which have enough resolution to detect non-Gaussianity of CMB anisotropy data with high precision [27].

Since a positive future detection of non-linearity in the CMB anisotropy pattern might allow to discriminate among the mechanisms by which cosmological adiabatic perturbations are generated, it is clear that the precise determination of the non-Gaussianity predicted by the various mechanisms is of primary interest.

In this paper we achieve different goals:

- We present a gauge-invariant formalism at second-order to study the evolution of curvature perturbations in a Friedmann-Robertson-Walker universe filled by multiple interacting fluids. We apply this general formalism to describe the evolution of the second-order gauge invariant curvature perturbations from inflation down to the radiation/matter phase in the various scenarios for the generation of the cosmological perturbations. In particular, we study the evolution of the total curvature perturbation during the non-adiabatic phase when either the inflaton field (for the standard and the inhomogeneous reheating scenario) or the curvaton field oscillate and decay into radiation. We show that the total curvature perturbation is exactly conserved in the standard scenario during the reheating phase after inflation and is sourced by a non-adiabatic pressure in the curvaton and the inhomogeneous scenarios.

- We present the exact expression for the second-order temperature fluctuations on super-horizon scales, extending the well-known expression for the Sachs-Wolfe effect at linear order.

- We provide the exact definition for the non-linear parameter $f_{\rm NL}$ entering in the determination of higher-order statistics as the bispectrum of the temperature anisotropies.

- We compute the exact expressions for the non-linear parameter $f_{\rm NL}$ in the different scenarios for the creation of cosmological perturbations.

The paper is organized as follows. In Section II we study the time-evolution for the system composed by a generic scalar field decaying into radiation which can be used to study a reheating phase. In Section III we describe in a gauge-invariant manner the evolution of cosmological perturbations at linear order while in Section IV we extend this analysis to second-order providing the evolution equations for the curvature perturbations. In Section V we apply our findings to the various scenarios for the generation of the cosmological perturbations and give the exact expressions for the second-order total curvature perturbation. In Section VI we compute the expression for the second-order temperature fluctuations on large scales and the corresponding non-linear parameter. In Section VII the predictions of the non-linear parameter for the various scenarios are provided. Finally, our conclusions are contained in Section VIII.

II. THE METRIC AND THE BACKGROUND EQUATIONS

In this section we provide the equations describing the time-evolution for the system composed by a generic scalar field and radiation. In particular, we will focus on the evolution of cosmological perturbations on large scales, up to second-order, for a system composed by a scalar field oscillating around the minimum of its potential and a radiation fluid having in mind the physical case of reheating. Averaged over several oscillations the effective equation of state of the scalar field φ is $w_\varphi = \langle P_\varphi / \rho_\varphi \rangle = 0$, where P_φ and ρ_φ are the scalar field pressure and energy density respectively. The scalar field is thus equivalent to a fluid of non-relativistic particles [28]. Moreover it is supposed to decay into radiation (light particles) with a decay rate Γ. We can thus describe this system as a pressureless and a radiation fluid which interact through an energy transfer triggered by the decay rate Γ. We follow the gauge-invariant approach developed in Ref. [25] to study cosmological perturbations at first-order for the general case of an arbitrary number of interacting fluids and we shall extend the analysis to second-order in the perturbations. Indeed the system under study encompasses the dynamics of the three main mechanisms for the generation of the primordial cosmological density perturbations on large scales, namely the standard scenario of single field inflation [1,2], the curvaton scenario [13–16], and the recently introduced scenario of "inhomogeneous reheating" [17–19,29]. In particular our second-order results allow us to calculate the level of non-Gaussianity in the gravitational potential in the latter scenario, where the inflaton decay rate has spatial fluctuations $\delta\Gamma$, produced when the couplings of the inflaton field to normal matter depends on the vacuum expectations value of some other light scalar fields χ present during inflation.

Let us consider the system composed by the oscillating scalar field φ and the radiation fluid. Each component has energy-momentum tensor $T^{\mu\nu}_{(\varphi)}$ and $T^{\mu\nu}_{(\gamma)}$. The total

energy momentum $T^{\mu\nu} = T^{\mu\nu}_{(\varphi)} + T^{\mu\nu}_{(\gamma)}$ is covariantly conserved, but allowing for an interaction between the two fluids [30]

$$\nabla_\mu T^{\mu\nu}_{(\varphi)} = Q^\nu_{(\varphi)} \,, \qquad \nabla_\mu T^{\mu\nu}_{(\gamma)} = Q^\nu_{(\gamma)} \,, \tag{1}$$

where $Q^\nu_{(\varphi)}$ and $Q^\nu_{(\gamma)}$ are the generic energy-momentum transfer to the scalar field and radiation sector respectively and are subject to the constraint

$$Q^\nu_{(\varphi)} + Q^\nu_{(\gamma)} = 0 \,. \tag{2}$$

The energy-momentum transfer $Q^\nu_{(\varphi)}$ and $Q^\nu_{(\gamma)}$ can be decomposed for convenience as [30]

$$Q^\nu_{(\varphi)} = \hat{Q}_\varphi u^\nu + f^\nu_{(\varphi)} \,, \qquad Q^\nu_{(\gamma)} = \hat{Q}_\gamma u^\nu + f^\nu_{(\gamma)} \,, \tag{3}$$

where the f^ν's are required to be orthogonal to the the total velocity of the fluid u^ν. The energy continuity equations for the scalar field and radiation can be obtained from $u_\nu \nabla_\mu T^{\mu\nu}_{(\varphi)} = u_\nu Q^\nu_{(\varphi)}$ and $u_\nu \nabla_\mu T^{\mu\nu}_{(\gamma)} = u_\nu Q^\nu_{(\gamma)}$ and hence from Eq. (3)

$$u_\nu \nabla_\mu T^{\mu\nu}_{(\varphi)} = \hat{Q}_\varphi \,, \qquad u_\nu \nabla_\mu T^{\mu\nu}_{(\gamma)} = \hat{Q}_\gamma \,. \tag{4}$$

In the case of an oscillating scalar field decaying into radiation the energy transfer coefficient $\hat{Q}_\varphi$ is given by [20]

$$\hat{Q}_\varphi = -\Gamma \rho_\varphi \qquad \hat{Q}_\gamma = \Gamma \rho_\varphi \;,, \tag{5}$$

where Γ is the decay rate of the scalar field into radiation.

A consistent study of the density perturbations requires to consider the metric perturbations around a spatially-flat Friedmann-Robertson-Walker (FRW) background as well. In the following we have used the expression for the metric perturbations contained in Refs. [31,32]. The components of a perturbed spatially flat Robertson-Walker metric can be written as

$$\begin{aligned} g_{00} &= -a^2(\tau)\left(1 + 2\phi^{(1)} + \phi^{(2)}\right) , \\ g_{0i} &= a^2(\tau)\left(\hat{\omega}_i^{(1)} + \frac{1}{2}\hat{\omega}_i^{(2)}\right) , \\ g_{ij} &= a^2(\tau)\left[(1 - 2\psi^{(1)} - \psi^{(2)})\delta_{ij} + \left(\hat{\chi}_{ij}^{(1)} + \frac{1}{2}\hat{\chi}_{ij}^{(2)}\right)\right] , \end{aligned} \tag{6}$$

where $a(\tau)$ is the scale factor and $\tau = \int dt/a$ is the conformal time. The standard splitting of the perturbations into scalar, transverse (*i.e* divergence-free) vector parts, and transverse trace-free tensor parts with respect to the 3-dimensional space with metric δ_{ij} can be performed in the following way:

$$\hat{\omega}_i^{(r)} = \partial_i \omega^{(r)} + \omega_i^{(r)} \,, \tag{7}$$

$$\hat{\chi}_{ij}^{(r)} = D_{ij}\chi^{(r)} + \partial_i \chi_j^{(r)} + \partial_j \chi_i^{(r)} + \chi_{ij}^{(r)} \,, \tag{8}$$

where $r = 1, 2$ stand for the order of the perturbations, ω_i and χ_i are transverse vectors ($\partial^i \omega_i^{(r)} = \partial^i \chi_i^{(r)} = 0$), $\chi_{ij}^{(r)}$ is a symmetric transverse and trace-free tensor ($\partial^i \chi_{ij}^{(r)} = 0$, $\chi^{i(r)}_{\ i} = 0$) and $D_{ij} = \partial_i \partial_j - (1/3)\, \delta_{ij}\, \partial^k \partial_k$ is a trace-free operator. Here and in the following latin indices are raised and lowered using δ^{ij} and δ_{ij}, respectively.
For our purposes the metric in Eq. (6) can be simplified. In fact, first-order vector perturbations are zero; moreover, the tensor part gives a negligible contribution to second-order perturbations. Thus, in the following we can neglect $\omega_i^{(1)}$, $\chi_i^{(1)}$ and $\chi_{ij}^{(1)}$. However the same is not true for the second-order perturbations. In the second-order theory the second-order vector and tensor contributions can be generated by the first-order scalar perturbations even if they are initially zero [31]. Thus we have to take them into account and we shall use the metric

$$\begin{aligned} g_{00} &= -a^2(\tau)\left(1 + 2\phi^{(1)} + \phi^{(2)}\right) , \\ g_{0i} &= a^2(\tau)\left(\partial_i \omega^{(1)} + \frac{1}{2}\,\partial_i \omega^{(2)} + \frac{1}{2}\,\omega_i^{(2)}\right) , \\ g_{ij} &= a^2(\tau)\left[\left(1 - 2\psi^{(1)} - \psi^{(2)}\right)\delta_{ij} + D_{ij}\left(\chi^{(1)} + \frac{1}{2}\chi^{(2)}\right) \right. \\ &\quad \left. + \frac{1}{2}\left(\partial_i \chi_j^{(2)} + \partial_j \chi_i^{(2)} + \chi_{ij}^{(2)}\right)\right] . \end{aligned} \tag{9}$$

The controvariant metric tensor is obtained by requiring (up to second-order) that $g_{\mu\nu} g^{\nu\lambda} = \delta_\mu^{\ \lambda}$ and it is given by

$$\begin{aligned} g^{00} &= -a^{-2}(\tau)\left(1 - 2\phi^{(1)} - \phi^{(2)} + 4\left(\phi^{(1)}\right)^2 \right. \\ &\quad \left. - \partial^i \omega^{(1)} \partial_i \omega^{(1)}\right) , \\ g^{0i} &= a^{-2}(\tau)\left[\partial^i \omega^{(1)} + \frac{1}{2}\left(\partial^i \omega^{(2)} + \omega^{i(2)}\right) \right. \\ &\quad \left. + 2\left(\psi^{(1)} - \phi^{(1)}\right)\partial^i \omega^{(1)} - \partial^i \omega^{(1)} D^i_{\ k} \chi^{(1)}\right] , \\ g^{ij} &= a^{-2}(\tau)\left[\left(1 + 2\psi^{(1)} + \psi^{(2)} + 4\left(\psi^{(1)}\right)^2\right)\delta^{ij} \right. \\ &\quad - D^{ij}\left(\chi^{(1)} + \frac{1}{2}\chi^{(2)}\right) - \frac{1}{2}\left(\partial^i \chi^{j(2)} + \partial^j \chi^{i(2)} + \chi^{ij(2)}\right) \\ &\quad \left. - \partial^i \omega^{(1)} \partial^j \omega^{(1)} - 4\psi^{(1)} D^{ij} \chi^{(1)} + D^{ik} \chi^{(1)} D^j_{\ k} \chi^{(1)}\right] . \end{aligned} \tag{10}$$

A. Background equations

The evolution of the FRW background universe is governed by the Friedmann constraint

$$\mathcal{H}^2 = \frac{8\pi G}{3}\rho a^2\,, \tag{11}$$

and the continuity equation

$$\rho' = -3\mathcal{H}\left(\rho + P\right)\,, \tag{12}$$

where a prime denotes differentiation with respect to conformal time, $\mathcal{H} \equiv a'/a$ is the Hubble parameter in conformal time, and ρ and P are the total energy density and the total pressure of the system. The total energy density and the total pressure are related to the energy density and pressure of the scalar field and radiation by

$$\begin{aligned} \rho &= \rho_\varphi + \rho_\gamma\,, \\ P &= P_\varphi + P_\gamma\,, \end{aligned} \tag{13}$$

where P_γ is the radiation pressure. The energy continuity equations for the energy density of the scalar field ρ_φ and radiation ρ_γ in the background are

$$\rho'_\varphi = -3\mathcal{H}\left(\rho_\phi + P_\phi\right) + aQ_\varphi\,, \tag{14}$$

$$\rho'_\gamma = -4\mathcal{H}\left(\rho_\gamma + P_\gamma\right) + aQ_\gamma\,, \tag{15}$$

where Q_φ and Q_γ indicate the background values of the transfer coefficient $\hat{Q}_\varphi$ and $\hat{Q}_\gamma$, respectively.

III. GAUGE-INVARIANT PERTURBATIONS AT FIRST-ORDER

The primordial adiabatic density perturbation is associated with a perturbation in the spatial curvature ψ and it is usually characterized in a gauge-invariant manner by the curvature perturbation on hypersurfaces of uniform total density ρ, usually indicated with ζ. Indeed, both the density perturbations, $\delta\rho$, $\delta\rho_\phi$ and $\delta\rho_\gamma$, and the curvature perturbation, ψ, are in general gauge-dependent. Specifically they depend upon the chosen time-slicing in an inhomogeneous universe. The curvature perturbation on fixed time hypersurfaces is a gauge-dependent quantity: after an arbitrary linear coordinate transformation, $\tau \to \tau + \delta\tau$, it transforms at first-order as $\psi^{(1)} \to \psi^{(1)} + \mathcal{H}\delta\tau$. For a scalar quantity, such as the energy density, the corresponding transformation is, at first-order, $\delta^{(1)}\rho \to \delta^{(1)}\rho - \rho'\delta\tau$. However a gauge-invariant combination ζ can be constructed which describes the density perturbation on uniform curvature slices or, equivalently the curvature of uniform density slices [7]

$$\zeta^{(1)} = -\psi^{(1)} - \mathcal{H}\frac{\delta^{(1)}\rho}{\rho'}\,. \tag{16}$$

Similarly it is possible to define the curvature perturbations ζ_i associated with each individual energy density components, which to linear order are defined as [25]

$$\zeta^{(1)}_\varphi = -\psi^{(1)} - \mathcal{H}\left(\frac{\delta^{(1)}\rho_\varphi}{\rho'_\varphi}\right)\,, \tag{17}$$

$$\zeta^{(1)}_\gamma = -\psi^{(1)} - \mathcal{H}\left(\frac{\delta^{(1)}\rho_\gamma}{\rho'_\gamma}\right)\,. \tag{18}$$

Notice that the total curvature perturbation ζ can be expressed as a weighted sum of the single curvature perturbations of the scalar field and radiation fluid as

$$\zeta^{(1)} = f\zeta^{(1)}_\varphi + (1-f)\zeta^{(1)}_\gamma\,. \tag{19}$$

where

$$f = \frac{\rho'_\varphi}{\rho'}\,, \quad 1 - f = \frac{\rho'_\gamma}{\rho'} \tag{20}$$

define the contribution of the scalar field and radiation to the total curvature perturbation $\zeta^{(1)}$, respectively. On the other hand the relative energy density perturbations can be characterized in a gauge-invariant manner through the difference between the two curvature perturbations

$$\mathcal{S}^{(1)}_{\phi\gamma} = 3(\zeta^{(1)}_\phi - \zeta^{(1)}_\gamma) = -3\mathcal{H}\left(\frac{\delta^{(1)}\rho_\phi}{\rho'_\phi} - \frac{\delta^{(1)}\rho_\gamma}{\rho'_\gamma}\right)\,. \tag{21}$$

The quantity $\mathcal{S}^{(1)}_{\phi\gamma}$ is usually called the entropy or isocurvature perturbation [30].

A. Evolution of first-order curvature perturbations on large scales

The equations of motion for the curvature perturbations $\zeta^{(1)}_\varphi$ and $\zeta^{(1)}_\gamma$ can be obtained perturbing at first order the continuity energy equations (4) for the scalar field and radiation energy densities, including the energy transfer. Expanding the transfer coefficients $\hat{Q}_\varphi$ and $\hat{Q}_\gamma$ up to first order in the perturbations around the homogeneous background as

$$\hat{Q}_\varphi = Q_\varphi + \delta^{(1)}Q_\varphi\,, \tag{22}$$

$$\hat{Q}_\gamma = Q_\gamma + \delta^{(1)}Q_\gamma\,, \tag{23}$$

Eqs. (4) give – on wavelengths larger than the horizon scale – [†]

$$\begin{aligned} &\delta^{(1)}\rho'_\varphi + 3\mathcal{H}\left(\delta^{(1)}\rho_\varphi + \delta^{(1)}P_\varphi\right) - 3\left(\rho_\varphi + P_\varphi\right)\psi^{(1)'} \\ &= a\,Q_\varphi\phi^{(1)} + a\,\delta^{(1)}Q_\varphi\,, \end{aligned} \tag{24}$$

$$\begin{aligned} &\delta^{(1)}\rho'_\gamma + 3\mathcal{H}\left(\delta^{(1)}\rho_\gamma + \delta^{(1)}P_\gamma\right) - 3\left(\rho_\gamma + P_\gamma\right)\psi^{(1)'} \\ &= a\,Q_\gamma\phi^{(1)} + a\,\delta^{(1)}Q_\gamma\,. \end{aligned} \tag{25}$$

[†] Here and in the following , as in Refs. [33,35], we neglect gradient terms which, upon integration over time, may give rise to non-local operators. However, note that these gradient terms will not affect the gravitational potential bispectrum on large scales.

Notice that the oscillating scalar field and radiation have fixed equations of state with $\delta^{(1)}P_\varphi = 0$ and $\delta^{(1)}P_\gamma = \delta^{(1)}\rho_\gamma/3$ (which correspond to vanishing intrinsic non-adiabatic pressure perturbations). Using the perturbed $(0-0)$-component of Einstein's equations for super-horizon wavelengths $\psi^{(1)'} + \mathcal{H}\phi^{(1)} = -\frac{\mathcal{H}}{2}\frac{\delta^{(1)}\rho}{\rho}$, we can rewrite Eqs. (24) and (25) in terms of the gauge-invariant curvature perturbations $\zeta_\varphi^{(1)}$ and $\zeta_\gamma^{(1)}$ [25]

$$
\begin{aligned}
\zeta_\varphi^{(1)'} &= \frac{a\mathcal{H}}{\rho'_\varphi}\Bigg[\delta^{(1)}Q_\varphi - \frac{Q'_\varphi}{\rho'_\varphi}\delta^{(1)}\rho_\varphi \\
&\quad + Q_\varphi\frac{\rho'}{2\rho}\left(\frac{\delta^{(1)}\rho_\varphi}{\rho'_\varphi} - \frac{\delta^{(1)}\rho}{\rho'}\right)\Bigg]\,, \qquad (26)\\
\zeta_\gamma^{(1)'} &= \frac{a\mathcal{H}}{\rho'_\gamma}\Bigg[\delta^{(1)}Q_\gamma - \frac{Q'_\gamma}{\rho'_\gamma}\delta^{(1)}\rho_\gamma \\
&\quad + Q_\gamma\frac{\rho'}{2\rho}\left(\frac{\delta^{(1)}\rho_\gamma}{\rho'_\gamma} - \frac{\delta^{(1)}\rho}{\rho'}\right)\Bigg]\,, \qquad (27)
\end{aligned}
$$

where $\delta^{(1)}Q_\gamma = -\delta^{(1)}Q_\varphi$ from the constraint in Eq (2). If the energy transfer coefficients $\hat{Q}_\varphi$ and $\hat{Q}_\gamma$ are given in terms of the decay rate Γ as in Eq. (5), the first order perturbation are respectively

$$
\delta^{(1)}Q_\varphi = -\Gamma\delta^{(1)}\rho_\varphi - \delta^{(1)}\Gamma\,\rho_\varphi\,, \qquad (28)
$$

$$
\delta^{(1)}Q_\gamma = \Gamma\delta^{(1)}\rho_\varphi + \delta^{(1)}\Gamma\,\rho_\varphi\,, \qquad (29)
$$

where notice in particular that we have allowed for a perturbation in the decay rate Γ,

$$
\Gamma(\tau,\mathbf{x}) = \Gamma(\tau) + \delta^{(1)}\Gamma(\tau,\mathbf{x})\,. \qquad (30)
$$

Such perturbations in the inflaton decay rate are indeed the key feature of the "inhomogeneous reheating" scenario [17–19]. In fact from now on we shall consider the background value Γ of the decay rate as constant in time, $\Gamma \approx \Gamma_*$ as this is the case for the the standard case of inflation and the inhomogeneous reheating mechanism. In such a case $\delta^{(1)}\Gamma$ is automatically gauge-invariant (for a gauge-invariant generalization in the case of $\Gamma' \neq 0$, see Ref. [29]). Plugging the expressions (28-29) into Eqs. (26-27), and using Eq. (19), the first order curvature perturbations for the scalar field and radiation obey on large scales

$$
\zeta_\varphi^{(1)'} = \frac{a\Gamma}{2}\frac{\rho_\varphi}{\rho'_\varphi}\frac{\rho'}{\rho}\left(\zeta^{(1)} - \zeta_\varphi^{(1)}\right) + a\mathcal{H}\frac{\rho_\varphi}{\rho'_\varphi}\delta^{(1)}\Gamma\,, \qquad (31)
$$

$$
\zeta_\gamma^{(1)'} = -\frac{a}{\rho'_\gamma}\left[\Gamma\rho'\frac{\rho'_\varphi}{\rho'_\gamma}\left(1 - \frac{\rho_\varphi}{2\rho}\right)\left(\zeta^{(1)} - \zeta_\varphi^{(1)}\right) + \mathcal{H}\rho_\varphi\delta^{(1)}\Gamma\right]\,. \qquad (32)
$$

From Eq. (19) it is thus possible to find the equation of motion for the total curvature perturbation $\zeta^{(1)}$ using the evolution of the individual curvature perturbations in Eqs. (31) and (32)

$$
\begin{aligned}
\zeta^{(1)'} &= f'\left(\zeta_\varphi^{(1)} - \zeta_\gamma^{(1)}\right) + f\zeta_\varphi^{(1)'} + (1-f)\zeta_\gamma^{(1)'} \\
&= \mathcal{H}f(1-f)\left(\zeta_\varphi^{(1)} - \zeta_\gamma^{(1)}\right) = -\mathcal{H}f\left(\zeta^{(1)} - \zeta_\varphi^{(1)}\right)\,. \qquad (33)
\end{aligned}
$$

Notice that during the decay of the scalar field into the radiation fluid, ρ'_γ in Eq. (32) may vanish. So it is convenient to close the system of equations by using the two equations (31) and (33) for the evolution of $\zeta_\varphi^{(1)}$ and $\zeta^{(1)}$.

IV. SECOND-ORDER CURVATURE PERTURBATIONS

We now generalize to second-order in the density perturbations the results of the previous section. In particular we obtain an equation of motion on large scales for the individual second-order curvature perturbations which include also the energy transfer between the scalar field and the radiation component.

As it has been shown in Ref. [33] it is possible to define the second-order curvature perturbation on uniform total density hypersurfaces by the quantity (up to a gradient term)

$$
\begin{aligned}
\zeta^{(2)} &= -\psi^{(2)} - \mathcal{H}\frac{\delta^{(2)}\rho}{\rho'} \\
&\quad + 2\mathcal{H}\frac{\delta^{(1)}\rho'}{\rho'}\frac{\delta^{(1)}\rho}{\rho'} + 2\frac{\delta^{(1)}\rho}{\rho'}\left(\psi^{(1)\prime} + 2\mathcal{H}\psi^{(1)}\right) \\
&\quad - \left(\frac{\delta^{(1)}\rho}{\rho'}\right)^2\left(\mathcal{H}\frac{\rho''}{\rho'} - \mathcal{H}' - 2\mathcal{H}^2\right)\,, \qquad (34)
\end{aligned}
$$

where the curvature perturbation ψ has been expanded up to second-order as $\psi = \psi^{(1)} + \frac{1}{2}\psi^{(2)}$ and $\delta^{(2)}\rho$ corresponds to the second-order perturbation in the total energy density around the homogeneous background $\rho(\tau)$

$$
\begin{aligned}
\rho(\tau,\mathbf{x}) &= \rho(\tau) + \delta\rho(\tau,\mathbf{x}) \\
&= \rho(\tau) + \delta^{(1)}\rho(\tau,\mathbf{x}) + \frac{1}{2}\delta^{(2)}\rho(\tau,\mathbf{x})\,. \qquad (35)
\end{aligned}
$$

The quantity $\zeta^{(2)}$ is gauge-invariant and, as its first-order counterpart defined in Eq. (16), it is sourced on super-horizon scales by a second-order non-adiabatic pressure perturbation [33].

In the standard scenario where the generation of cosmological perturbations is induced by fluctuations of a single inflaton field (and there is no curvaton) the evolution of the perturbations is purely adiabatic, and the total curvature perturbation $\zeta^{(2)}$ is indeed conserved. In Ref. [34] the conserved quantity $\zeta^{(2)}$ has been used to follow the evolution on large scales of the primordial non-linearity in the cosmological perturbations from a period of inflation to the matter dominated era. On the contrary in the curvaton and inhomogeneous reheating scenarios the total curvature perturbation $\zeta^{(2)}$ evolves on large scales due to a non-adiabatic pressure. In a similar manner to the linear order it is possible to follow

the evolution of $\zeta^{(2)}$ through the evolution of the density perturbations in the scalar field and radiation.

Let us introduce now the curvature perturbations $\zeta_i^{(2)}$ at second-order for each individual component. Such quantities will be given by the same formula as Eq. (34) relatively to each energy density ρ_i

$$\begin{aligned}\zeta_i^{(2)} &= -\psi^{(2)} - \mathcal{H}\frac{\delta^{(2)}\rho_i}{\rho_i'} \\ &+ 2\mathcal{H}\frac{\delta^{(1)}\rho_i'}{\rho_i'}\frac{\delta^{(1)}\rho_i}{\rho_i'} + 2\frac{\delta^{(1)}\rho_i}{\rho_i'}\left(\psi^{(1)\prime} + 2\mathcal{H}\psi^{(1)}\right) \\ &- \left(\frac{\delta^{(1)}\rho_i}{\rho_i'}\right)^2\left(\mathcal{H}\frac{\rho_i''}{\rho_i'} - \mathcal{H}' - 2\mathcal{H}^2\right). \end{aligned} \tag{36}$$

Since the quantities $\zeta_i^{(1)}$ and $\zeta_i^{(2)}$ are gauge-invariant, we choose to work in the spatially flat gauge $\psi^{(1)} = \psi^{(2)} = 0$ if not otherwise specified. Note that from Eqs. (17) and (18) $\zeta_\varphi^{(1)}$ and $\zeta_\gamma^{(1)}$ are thus given by

$$\zeta_\varphi^{(1)} = -\mathcal{H}\left(\frac{\delta^{(1)}\rho_\varphi}{\rho_\varphi'}\right), \tag{37}$$

$$\zeta_\gamma^{(1)} = -\mathcal{H}\left(\frac{\delta^{(1)}\rho_\gamma}{\rho_\gamma'}\right). \tag{38}$$

Using Eqs. (37) and (38), the energy continuity equations at first order (24) and (25) in the spatially flat gauge $\psi^{(1)} = 0$ yield

$$\begin{aligned}\frac{\delta^{(1)}\rho'}{\rho'} &= 3f\zeta_\varphi^{(1)} + 4(1-f)\zeta_\gamma^{(1)}, \\ \mathcal{H}\frac{\delta^{(1)}\rho}{\rho'} &= -f\zeta_\varphi^{(1)} - (1-f)\zeta_\gamma^{(1)}. \end{aligned} \tag{39}$$

We can thus rewrite the total second-order curvature perturbation $\zeta^{(2)}$ in Eq. (34) as

$$\begin{aligned}\zeta^{(2)} &= -\mathcal{H}\frac{\delta^{(2)}\rho}{\rho'} \\ &- \left[f\zeta_\varphi^{(1)} + (1-f)\zeta_\gamma^{(1)}\right]\left[f^2\zeta_\varphi^{(1)} + (1-f)(2+f)\zeta_\gamma^{(1)}\right], \end{aligned} \tag{40}$$

where we have used the background equations (14) and (15) to find $\mathcal{H}\frac{\rho''}{\rho'} - \mathcal{H}' - 2\mathcal{H}^2 = -\mathcal{H}^2(6-f)$.

Following the same procedure the individual curvature perturbations defined in Eq. (36) are given by

$$\begin{aligned}\zeta_\varphi^{(2)} &= -\mathcal{H}\frac{\delta^{(2)}\rho_\varphi}{\rho_\varphi'} + [2 - 3(1+w_\varphi)]\left(\zeta_\varphi^{(1)}\right)^2 \\ &- 2\left(a\frac{Q_\varphi\phi^{(1)}}{\rho_\varphi'} + a\frac{\delta^{(1)}Q_\varphi}{\rho_\varphi'}\right)\zeta_\varphi^{(1)} \\ &- \left[a\frac{Q_\varphi'}{\mathcal{H}\rho_\varphi'} - \frac{a}{2}\frac{Q_\varphi}{\mathcal{H}\rho_\varphi'}\frac{\rho'}{\rho}\right]\left(\zeta_\varphi^{(1)}\right)^2, \end{aligned} \tag{41}$$

$$\begin{aligned}\zeta_\gamma^{(2)} &= -\mathcal{H}\frac{\delta^{(2)}\rho_\gamma}{\rho_\gamma'} + [2 - 3(1+w_\gamma)]\left(\zeta_\gamma^{(1)}\right)^2 \\ &- 2\left(a\frac{Q_\gamma\phi^{(1)}}{\rho_\gamma'} + a\frac{\delta^{(1)}Q_\gamma}{\rho_\gamma'}\right)\zeta_\gamma^{(1)} \\ &- \left[a\frac{Q_\gamma'}{\mathcal{H}\rho_\gamma'} - \frac{a}{2}\frac{Q_\gamma}{\mathcal{H}\rho_\gamma'}\frac{\rho'}{\rho}\right]\left(\zeta_\gamma^{(1)}\right)^2, \end{aligned} \tag{42}$$

where $w_\gamma = 1/3$ is the radiation equation of state. Using Eqs. (41) and (42) to express the perturbation of the total energy density $\delta^{(2)}\rho$ one obtains the following expression for the total curvature perturbation $\zeta^{(2)}$

$$\begin{aligned}\zeta^{(2)} &= f\zeta_\varphi^{(2)} + (1-f)\zeta_\gamma^{(2)} \\ &+ f(1-f)(1+f)\left(\zeta_\varphi^{(1)} - \zeta_\gamma^{(1)}\right)^2 \\ &+ 2\left(a\frac{Q_\varphi\phi^{(1)}}{\rho'} + a\frac{\delta^{(1)}Q_\varphi}{\rho'}\right)\left[\zeta_\varphi^{(1)} - \zeta_\gamma^{(1)}\right] \\ &+ \left(a\frac{Q_\varphi'}{\mathcal{H}\rho'} - \frac{a}{2}\frac{Q_\varphi}{\mathcal{H}\rho}\right)\left[\left(\zeta_\varphi^{(1)}\right)^2 - \left(\zeta_\gamma^{(1)}\right)^2\right] \end{aligned} \tag{43}$$

The (0-0)-component of Einstein equations in the spatially flat gauge at first-order $\psi^{(1)} = 0$

$$\phi^{(1)} = -\frac{1}{2}\frac{\delta^{(1)}\rho}{\rho} = \frac{1}{2}\frac{\rho'}{\mathcal{H}\rho}\zeta^{(1)}, \tag{44}$$

and the explicit expressions of the first order perturbed coefficients in terms of the decay rate Γ, Eqs. (28-29), yield

$$\begin{aligned}\zeta^{(2)} &= f\zeta_\varphi^{(2)} + (1-f)\zeta_\gamma^{(2)} + f(1-f)(1+f)\left(\zeta_\varphi^{(1)} - \zeta_\gamma^{(1)}\right)^2 \\ &+ \frac{a\,\Gamma}{\mathcal{H}}f\left(\zeta_\varphi^{(1)} - \zeta_\gamma^{(1)}\right)^2 - 2a\delta^{(1)}\Gamma\frac{\rho_\varphi}{\rho'}\left(\zeta_\varphi^{(1)} - \zeta_\gamma^{(1)}\right) \\ &+ \frac{a\Gamma}{\mathcal{H}}(1-2f)\frac{\rho_\varphi}{2\rho}\left(\zeta_\varphi^{(1)} - \zeta_\gamma^{(1)}\right)^2. \end{aligned} \tag{45}$$

Eq. (45) is the generalization to second-order in the perturbations of the weighted sum in Eq. (19) and extends the expression found in Ref. [35] in the particular case of the curvaton scenario, under the sudden decay approximation, where the energy transfer was neglected.

A. Large-scale evolution of second-order curvature perturbations

In this section we give the equations of motion on large scales for the individual second-order curvature perturbations $\zeta_\varphi^{(2)}$ and $\zeta_\gamma^{(2)}$. The energy transfer coefficients $\hat{Q}_\varphi$ and $\hat{Q}_\gamma$ in Eqs. (3) perturbed at second-order around the homogeneous backgrounds are given by

$$\hat{Q}_\varphi = Q_\varphi + \delta^{(1)}Q_\varphi + \frac{1}{2}\delta^{(2)}Q_\varphi, \tag{46}$$

$$\hat{Q}_\gamma = Q_\gamma + \delta^{(1)}Q_\gamma, + \frac{1}{2}\delta^{(2)}Q_\gamma. \tag{47}$$

Note that from Eq. (2) it follows $\delta^{(2)}Q_\gamma = -\delta^{(2)}Q_\gamma$. Thus the energy continuity equations (4) perturbed at second-order give on large scales

$$\delta^{(2)}\rho_\varphi{}' + 3\mathcal{H}\left(\delta^{(2)}\rho_\varphi + \delta^{(2)}P_\varphi\right) - 3(\rho_\varphi + P_\varphi)\psi^{(2)'}$$
$$- 6\psi^{(1)'}\left[\delta^{(1)}\rho_\varphi + \delta^{(1)}P_\varphi + 2(\rho_\varphi + P_\varphi)\psi^{(1)}\right] =$$
$$a\ \delta^{(2)}Q_\varphi + a\,Q_\varphi\phi^{(2)} - aQ_\varphi\left(\phi^{(1)}\right)^2 + 2a\phi^{(1)}\delta^{(1)}Q_\varphi\,, \quad (48)$$

$$\delta^{(2)}\rho_\gamma{}' + 3\mathcal{H}\left(\delta^{(2)}\rho_\gamma + \delta^{(2)}P_\gamma\right) - 3(\rho_\gamma + P_\gamma)\psi^{(2)'}$$
$$- 6\psi^{(1)'}\left[\delta^{(1)}\rho_\gamma + \delta^{(1)}P_\gamma + 2(\rho_\gamma + P_\gamma)\psi^{(1)}\right] =$$
$$a\ \delta^{(2)}Q_\gamma + a\,Q_\gamma\phi^{(2)} - aQ_\gamma\left(\phi^{(1)}\right)^2 + 2a\phi^{(1)}\delta^{(1)}Q_\gamma\,, \quad (49)$$

where $\phi^{(2)}$ is the second-order perturbation in the gravitational potential $\phi = \phi^{(1)} + \frac{1}{2}\phi^{(2)}$. Note that Eqs. (48) and (49) hold in a generic gauge. We can now recast such equations in terms of the gauge-invariant curvature perturbations $\zeta^{(2)}_\varphi$ and $\zeta^{(2)}_\gamma$ in a straightforward way by choosing the spatially flat gauge $\psi^{(1)} = \psi^{(2)} = 0$.

The (0-0)-component of Einstein equations in the spatially flat gauge at first order is given by Eq. (44), and at second-order on large scales it reads (see Eqs. (A.39) in Ref. [32])

$$\phi^{(2)} = -\frac{1}{2}\frac{\delta^{(2)}\rho}{\rho} + 4\left(\phi^{(1)}\right)^2 . \quad (50)$$

Using Eqs. (44) and (50) with the expressions (41-42) we find from the energy continuity equations (48-49) that the individual second-order curvature perturbations obey on large scales

$$\zeta^{(2)'}_\varphi =$$
$$-\frac{a\mathcal{H}}{\rho'}\left[\left(\delta^{(2)}Q_\varphi - \frac{Q'_\varphi}{\rho'_\varphi}\delta^{(2)}\rho_\varphi\right) + Q_\varphi\frac{\rho'}{2\rho}\left(\frac{\delta^{(2)}\rho_\varphi}{\rho'_\varphi} - \frac{\delta^{(2)}\rho}{\rho'}\right)\right]$$
$$- 3aQ_\varphi\frac{\mathcal{H}}{\rho'_\varphi}\left(\phi^{(1)}\right)^2 - 2a\frac{\mathcal{H}}{\rho'_\varphi}\delta^{(1)}Q_\varphi\phi^{(1)} - 2\zeta^{(1)}_\varphi\zeta^{(1)'}_\varphi$$
$$- 2\left[\zeta^{(1)}_\varphi\left(a\frac{Q_\varphi\phi^{(1)}}{\rho'_\varphi} + a\frac{\delta^{(1)}Q_\varphi}{\rho'_\varphi}\right)\right]'$$
$$- \left[\left(\zeta^{(1)}_\varphi\right)^2\left(a\frac{Q'_\varphi}{\mathcal{H}\rho'_\varphi} - \frac{a}{2}\frac{Q_\varphi}{\mathcal{H}\rho'_\varphi}\frac{\rho'}{\rho}\right)\right]' \quad (51)$$

and

$$\zeta^{(2)'}_\gamma =$$
$$-\frac{a\mathcal{H}}{\rho'}\left[\left(\delta^{(2)}Q_\gamma - \frac{Q'_\gamma}{\rho'_\gamma}\delta^{(2)}\rho_\gamma\right) + Q_\gamma\frac{\rho'}{2\rho}\left(\frac{\delta^{(2)}\rho_\gamma}{\rho'_\gamma} - \frac{\delta^{(2)}\rho}{\rho'}\right)\right]$$
$$- 3aQ_\gamma\frac{\mathcal{H}}{\rho'_\gamma}\left(\phi^{(1)}\right)^2 - 2a\frac{\mathcal{H}}{\rho'_\gamma}\delta^{(1)}Q_\gamma\phi^{(1)} - 4\zeta^{(1)}_\gamma\zeta^{(1)'}_\gamma$$
$$- 2\left[\zeta^{(1)}_\gamma\left(a\frac{Q_\gamma\phi^{(1)}}{\rho'_\gamma} + a\frac{\delta^{(1)}Q_\gamma}{\rho'_\gamma}\right)\right]'$$
$$- \left[\left(\zeta^{(1)}_\gamma\right)^2\left(a\frac{Q'_\gamma}{\mathcal{H}\rho'_\gamma} - \frac{a}{2}\frac{Q_\gamma}{\mathcal{H}\rho'_\gamma}\frac{\rho'}{\rho}\right)\right]' \quad (52)$$

where we have used that $w_\varphi = 0$ and $w_\gamma = 1/3$.

Equations (51) and (52) allow to follow the time-evolution of the gauge-invariant curvature perturbation at second-order.

V. EVOLUTION OF SECOND-ORDER COSMOLOGICAL PERTURBATIONS IN THE VARIOUS SCENARIOS

The results contained in the previous section can be used in order to study the evolution of the second-order curvature perturbations during the reheating phase after a period of standard single field inflation, and in the alternative scenarios for the generation of the primordial adiabatic perturbations which have been recently proposed, namely the curvaton scenario [13–15] and the inhomogeneous reheating [17–19]. In fact in each of these scenarios a scalar field is oscillating around the minimum of its potential and eventually it decays into radiation. The evolution at second-order of the curvature perturbations is necessary in order to follow the non-linearity of the cosmological perturbations and thus to accurately compute the level of non-Gaussianity including all the relevant second-order effects. We shall first extend some of the results previously obtained in Ref. [35] for the standard scenario and the curvaton scenario. Then we will present the calculation of the second-order total curvature perturbation produced in the "inhomogeneous reheating" by the spatial fluctuations of the inflaton decay rate Γ.

A. Standard Scenario

In the standard inflationary scenario the observed density perturbations are due to the fluctuations of the inflaton field itself. When inflation ends, the inflaton oscillates about the minimum of its potential and decays, thereby reheating the universe. In such a standard scenario the inflaton decay rate has no spatial fluctuations. Here we want to show that in fact during the reheating phase the curvature perturbations $\zeta^{(1)}$ and $\zeta^{(2)}$ do remain constant on superhorizon scales. Eq. (33) and Eq. (31) with $\delta^{(1)}\Gamma = 0$ now read

$$\zeta^{(1)'} = -\mathcal{H}f\left(\zeta^{(1)} - \zeta^{(1)}_\varphi\right) , \quad (53)$$

$$\zeta^{(1)'}_\varphi = \frac{a\Gamma}{2}\frac{\rho_\varphi}{\rho'_\varphi}\frac{\rho'}{\rho}\left(\zeta^{(1)} - \zeta^{(1)}_\varphi\right) . \quad (54)$$

At the beginning of the reheating phase, after the end of inflation, the total curvature perturbation is initially

given by the curvature perturbation of the inflaton fluctuations $\zeta^{(1)}_{\rm in} = \zeta^{(1)}_{\varphi,{\rm in}}$. Therefore Eqs. (53) and (54) show that during the reheating phase $\zeta^{(1)} = \zeta^{(1)}_\varphi = \zeta^{(1)}_{\varphi,{\rm in}}$ is a fixed-point of the time-evolution. This result has been shown at first-order in [29] and extended to second-order in the perturbations in Ref. [35] under the sudden decay approximation. Under such an approximation the individual energy density perturbations (and hence the corresponding curvature perturbations) are separately conserved until the decay of the scalar field, which amounts to saying that in the equations for the curvature perturbations Eqs. (51) and (52) one can drop the energy transfer triggered by the decay rate $a\Gamma/\mathcal{H} \ll 1$. Going beyond the sudden decay approximation, the first order results $\zeta^{(1)}_\varphi = \zeta^{(1)}$ in Eq. (45) yield

$$\zeta^{(2)} = f\zeta^{(2)}_\varphi + (1-f)\zeta^{(2)}_\gamma \,. \tag{55}$$

Deriving this expression and using Eqs. (51) and (52) with $\delta^{(2)}\Gamma = 0$ and $\zeta^{(1)}_\varphi = \zeta^{(1)}$ the equation of motion for $\zeta^{(2)}$ on large scales reads

$$\zeta^{(2)\prime} = -\mathcal{H} f\left(\zeta^{(2)} - \zeta^{(2)}_\varphi\right) \,. \tag{56}$$

In the same way as at first order from Eqs. (55) and (56) it follows that the second-order curvature perturbation $\zeta^{(2)}$ remains constant on large scales during the reheating phase, being given at the end of inflation by the curvature perturbation in the inflaton field $\zeta^{(2)}_{\rm in} = \zeta^{(2)}_{\varphi,{\rm in}}$.

B. Curvaton scenario

The evolution of cosmological perturbations in the curvaton scenario has been studied at second-order in Ref. [35] and here we only briefly summarize the main results. In the curvaton scenario the final curvature perturbations are produced from an initial isocurvature perturbation associated to the quantum fluctuations of a light scalar field (other than the inflaton), the curvaton σ, whose energy density is negligible during inflation. The curvaton isocurvature perturbations are transformed into adiabatic ones when the curvaton decays into radiation much after the end of inflation [14,16]. During inflation, since no curvature perturbation is produced $\zeta^{(1)} = \zeta^{(2)} = 0$. After the end of inflation, the curvaton starts to oscillate when its mass is of the order of the Hubble rate and the first-order total curvature perturbation can be expressed as

$$\zeta^{(1)} = f\zeta^{(1)}_\sigma + (1-f)\zeta^{(1)}_\gamma \,, \tag{57}$$

where $f = \rho'_\sigma/\rho'$ and we have set $\zeta^{(1)}_\gamma = 0$ consistently with the curvaton hypothesis. When the curvaton energy density is subdominant, the density perturbation in the curvaton field $\zeta^{(1)}_\sigma$ gives a negligible contribution to the total curvature perturbation, thus corresponding to an isocurvature (or entropy) perturbation. On the other hand during the oscillations $\rho_\sigma \propto a^{-3}$ increases with respect to the energy density of radiation $\rho_\gamma \propto a^{-4}$, and the perturbations in the curvaton field are then converted into the curvature perturbation. After the decay of the curvaton, during the conventional radiation and matter dominated eras the total curvature perturbation will remain constant on superhorizon scales at a value which, in the sudden decay approximation, is given by

$$\zeta^{(1)} = f_D\, \zeta^{(1)}_\sigma \,, \tag{58}$$

where D stands for the epoch of the curvaton decay. Going beyond the sudden decay approximation it is possible to introduce a transfer parameter r defined as [16,25]

$$\zeta^{(1)} = r\zeta^{(1)}_\sigma \,, \tag{59}$$

where $\zeta^{(1)}$ is evaluated well after the epoch of the curvaton decay and $\zeta^{(1)}_\sigma$ is evaluated well before this epoch. The numerical study of the coupled perturbation equations has been performed in Ref. [25] showing that the sudden decay approximation is exact when the curvaton dominates the energy density before it decays ($r = 1$), while in the opposite case

$$r \approx \left(\frac{\rho_\sigma}{\rho}\right)_D \,. \tag{60}$$

At second-order, using Eqs. (51) and (52) under the sudden-decay approximation, the individual curvature perturbations $\zeta^{(2)}_\sigma$ and $\zeta^{(2)}_\gamma$ are separately conserved and the total curvature perturbation $\zeta^{(2)}$ reads [35]

$$\begin{aligned}\zeta^{(2)} &= f\zeta^{(2)}_\sigma + (1-f)\zeta^{(2)}_\gamma \\ &\quad + f(1-f)(1+f)\left(\zeta^{(1)}_\sigma - \zeta^{(1)}_\gamma\right)^2 \,.\end{aligned} \tag{61}$$

The second-order curvature perturbation in the standard radiation or matter eras will remain constant on superhorizon scales and, in the sudden decay approximation, it is thus given by the quantity in Eq. (61) evaluated at the epoch of the curvaton decay

$$\zeta^{(2)} = r\zeta^{(2)}_\sigma + r\left(1-r^2\right)\left(\zeta^{(1)}_\sigma\right)^2 \,, \tag{62}$$

where we have used the curvaton hypothesis that the curvature perturbation in the radiation produced at the end of inflation is negligible so that $\zeta^{(1)}_\gamma \approx 0$ and $\zeta^{(2)}_\gamma \approx 0$. Taking into account that $\zeta^{(2)}_\sigma = \frac{1}{2}\left(\zeta^{(1)}_\sigma\right)^2$ [35], one finally obtains the curvature perturbation during the standard radiation or matter dominated eras

$$\zeta^{(2)} = r\left(\frac{3}{2} - r^2\right)\left(\zeta^{(1)}_\sigma\right)^2 \,. \tag{63}$$

C. Inhomogeneous reheating Scenario: $\delta\Gamma \neq 0$

Recently, another mechanism for the generation of cosmological perturbations has been proposed [17–19]. It acts during the reheating stage after inflation and it was dubbed the "inhomogeneous reheating" mechanism in Ref. [19]. The coupling of the inflaton to normal matter may be determined by the vacuum expectation value of fields χ's of the underlying theory. If those fields are light during inflation, fluctuations $\delta\chi \sim \frac{H}{2\pi}$, where H is the Hubble rate during inflation, are left imprinted on super-horizon scales. These perturbations lead to spatial fluctuations in the decay rate Γ of the inflaton field to ordinary matter, $\frac{\delta\Gamma}{\Gamma} \sim \frac{\delta\chi}{\chi}$, causing adiabatic perturbations in the final reheating temperature in different regions of the universe. Using cosmic time $dt = a\, d\tau$ the first order equation (31) for $\zeta_\varphi^{(1)}$ on large scales now reads

$$\dot{\zeta}_\varphi^{(1)} = \frac{\Gamma}{2}\frac{\rho_\varphi}{\dot{\rho_\varphi}}\frac{\dot{\rho}}{\rho}\left(\zeta^{(1)} - \zeta_\varphi^{(1)}\right) + H\frac{\rho_\varphi}{\dot{\rho_\varphi}}\delta^{(1)}\Gamma\,, \tag{64}$$

where $H = \dot{a}/a$ is the Hubble parameter in cosmic time and the dots stand for differentation with respect to cosmic time. Perturbing the energy transfer $\hat{Q}_\varphi = -\Gamma\rho_\varphi$ up to second-order and expanding the decay rate as

$$\Gamma = \Gamma_* + \delta\Gamma = \Gamma_* + \delta^{(1)}\Gamma + \frac{1}{2}\delta^{(2)}\Gamma\,, \tag{65}$$

it follows from Eqs. (46) and (65)

$$\delta^{(2)}Q_\varphi = -\rho_\varphi\delta^{(2)}\Gamma - \Gamma_*\delta^{(2)}\rho_\varphi - 2\delta^{(1)}\Gamma\delta^{(1)}\rho_\varphi\,. \tag{66}$$

Plugging Eq. (66) into Eq. (51), the equation of motion on large scales for the curvature perturbation $\zeta_\varphi^{(2)}$ allowing for possible fluctuations of the decay rate $\delta^{(1)}\Gamma$ and $\delta^{(2)}\Gamma$ turns out to be

$$\begin{aligned}\dot{\zeta}_\varphi^{(2)} &= \frac{H}{\dot{\rho_\varphi}}\left(\delta^{(2)}\Gamma\rho_\varphi + 2\delta^{(1)}\Gamma\delta^{(1)}\rho_\varphi\right)\\ &- \frac{\Gamma_*\rho_\varphi}{2}\frac{\dot{\rho}}{\rho}\left(\frac{\delta^{(2)}\rho_\varphi}{\dot{\rho_\varphi}} - \frac{\delta^{(2)}\rho}{\dot{\rho}}\right) + 3\Gamma_*\rho_\varphi\frac{H}{\dot{\rho_\varphi}}{\phi^{(1)}}^2\\ &+ \frac{2H}{\dot{\rho_\varphi}}\left(\delta^{(1)}\Gamma\rho_\varphi + \delta^{(1)}\rho_\varphi\Gamma_*\right)\phi^{(1)} - 2\zeta_\varphi^{(1)}\dot{\zeta}_\varphi^{(1)}\\ &+ 2\left[\zeta_\varphi^{(1)}\left(\Gamma_*\frac{\rho_\varphi}{\dot{\rho_\varphi}}\phi^{(1)} + \delta^{(1)}\Gamma\frac{\rho_\varphi}{\dot{\rho_\varphi}} + \Gamma_*\frac{\delta^{(1)}\rho_\varphi}{\dot{\rho_\varphi}}\right)\right]^{\cdot}\\ &+ \left[{\zeta_\varphi^{(1)}}^2\frac{\Gamma}{H}\left(1 - \frac{\rho_\varphi}{\dot{\rho_\varphi}}\frac{\dot{\rho}}{\rho}\right)\right]^{\cdot}\,,\end{aligned} \tag{67}$$

where we have used the fact that the decay rate Γ in the scenario under consideration remains constant.

We shall now solve Eqs. (64) and (67) under a "mixed-sudden decay approximation". We shall treat the pressureless scalar field fluid and radiation fluid as if they are not interacting until the decay of the inflaton when $\Gamma \approx H$. Since at the beginning of the reheating phase the energy density in radiation is negligible this means that $f = \dot{\rho_\varphi}/\dot{\rho} \approx 1$ and there is indeed only a single fluid with, from Eq. (19), $\zeta^{(1)} \approx \zeta_\varphi^{(1)}$ and $\zeta_\gamma^{(1)} \approx 0$. In fact under such an approximation we can neglect all the terms proportional to the decay rate Γ, *but* we allow for the spatial fluctuations of the decay rate. Thus the first order equation (64) reads

$$\dot{\zeta}_\varphi^{(1)} \simeq -\frac{1}{3}\delta^{(1)}\Gamma\,, \tag{68}$$

where we have used $\dot{\rho_\varphi} = -3H\rho_\varphi$ in the sudden decay approximation. Integration over time yields

$$\zeta_\varphi^{(1)} = -\frac{t}{3}\delta^{(1)}\Gamma = -\frac{2}{9}\frac{\delta^{(1)}\Gamma}{H} \simeq \zeta^{(1)}\,, \tag{69}$$

where we have used that during the oscillations of the scalar field which dominates the energy density $H = \frac{2}{3t}$. The inhomogeneous reheating mechanism produces at linear level a gravitational potential which after the reheating phase, in the radiation dominated epoch, is given by (in the longitudinal gauge) [17]

$$\psi^{(1)} = \frac{1}{9}\frac{\delta^{(1)}\Gamma}{\Gamma_*}\,. \tag{70}$$

During the radiation dominated epoch the usual relation between the Bardeen potential and the curvature perturbation is $\psi^{(1)} = -\frac{2}{3}\zeta^{(1)}$, and thus from Eq. (69) we can set the ratio $\Gamma_*/H_D = \frac{3}{4}$ at the time of the inflaton decay in order to reproduce the result (70) of Ref. [17]. Thus from Eq. (69) it follows that the value of $\zeta^{(1)}$ is

$$\zeta^{(1)} \simeq -\frac{1}{6}\frac{\delta^{(1)}\Gamma}{\Gamma_*}\,. \tag{71}$$

We can use this result in order to solve the second-order equation (67), which under the sudden decay approximation and using Eq. (44) reads

$$\dot{\zeta}_\varphi^{(2)} \simeq -\frac{1}{3}\delta^{(2)}\Gamma - \zeta_\varphi^{(1)}\delta^{(1)}\Gamma - 2\left(\zeta_\varphi^{(1)}\dot{\zeta}_\varphi^{(1)}\right) - \frac{2}{3}\left(\frac{\delta^{(1)}\Gamma}{H}\zeta_\varphi^{(1)}\right)^{\cdot}\,. \tag{72}$$

The fluctuations $\delta\Gamma = \delta^{(1)}\Gamma + \frac{1}{2}\delta^{(2)}\Gamma$ indeed depends on the underlying physics for the coupling of the inflaton field to the other scalar field(s) χ. Let us take for example $\Gamma(t, \mathbf{x}) \propto \chi^2(t, \mathbf{x})$. If the scalar field χ is very light, its homogeneous value can be treated as constant $\chi(t) \approx \chi_*$ and during inflation quantum fluctuations $\delta^{(1)}\chi$ around its homogeneous value χ_* are left imprinted on super-horizon scales. Therefore non-linear fluctuations $\left(\delta^{(1)}\chi\right)^2$ of the decay rate Γ are produced as well

$$\Gamma \propto \chi^2 = \chi_*^2 + 2\chi_*\delta^{(1)}\chi + \left(\delta^{(1)}\chi\right)^2\,. \tag{73}$$

From Eqs. (65) and (73) it follows

$$\frac{\delta^{(1)}\Gamma}{\Gamma_*} = 2\frac{\delta^{(1)}\chi}{\chi_*}\,,$$
$$\frac{\delta^{(2)}\Gamma}{\Gamma_*} = 2\left(\frac{\delta^{(1)}\chi}{\chi_*}\right)^2 = \frac{1}{2}\left(\frac{\delta^{(1)}\Gamma}{\Gamma_*}\right)^2\,. \tag{74}$$

Using the first order solution $\zeta_\varphi^{(1)} = -\frac{t}{3}\delta^{(1)}\Gamma$ and Eq. (74) in Eq. (72), the evolution of $\zeta_\varphi^{(2)}$ on large scales is

$$\dot\zeta_\varphi^{(2)} \simeq -\frac{1}{6\Gamma_*}\left(\delta^{(1)}\Gamma\right)^2 + \frac{1}{3}\left(\delta^{(1)}\Gamma\right)^2 t - 2\left(\zeta_\varphi^{(1)}\dot\zeta_\varphi^{(1)}\right) - \frac{2}{3}\left(\frac{\delta^{(1)}\Gamma}{H}\zeta_\varphi^{(1)}\right)^{\cdot}\,. \tag{75}$$

Integration over time yields

$$\zeta_\varphi^{(2)} = -\frac{t}{6}\frac{\left(\delta^{(1)}\Gamma\right)^2}{\Gamma_*} + \frac{1}{6}\left(\delta^{(1)}\Gamma\right)^2 t^2 - \left(\zeta_\varphi^{(1)}\right)^2 - \frac{2}{3}\zeta_\varphi^{(1)}\frac{\delta^{(1)}\Gamma}{H}\,. \tag{76}$$

At the time of inflaton decay $\Gamma/H_D = \frac{3}{4}$, and since $H = 2/3\,t$, it follows $t_D = 1/2\Gamma_*$. Thus $\zeta_\varphi^{(2)}$ in Eq. (76) evaluated at the time t_D of inflaton decay is

$$\zeta_\varphi^{(2)} \simeq -\frac{1}{24}\left(\frac{\delta^{(1)}\Gamma}{\Gamma_*}\right)^2 - \zeta_\varphi^{(1)2} - \frac{1}{2}\zeta_\varphi^{(1)}\frac{\delta^{(1)}\Gamma}{\Gamma_*}\,. \tag{77}$$

Using Eq. (71) we finally find that the total curvature perturbation $\zeta^{(2)}$ in the sudden decay approximation is given by

$$\zeta^{(2)} \simeq \zeta_\varphi^{(2)} \simeq \frac{1}{2}\left(\zeta^{(1)}\right)^2\,. \tag{78}$$

VI. SECOND-ORDER TEMPERATURE FLUCTUATIONS ON LARGE SCALES

The goal of this section is to provide the expression for the second-order temperature fluctuations on large scales which will allow the exact definition of the non-linear parameter $f_{\rm NL}$. From now on, we will adopt the *Poisson gauge* [36] which is defined by the condition $\omega = \chi = \chi_i = 0$. Then, one scalar degree of freedom is eliminated from g_{0i} and one scalar and two vector degrees of freedom from g_{ij}. This gauge generalizes the so-called longitudinal gauge to include vector and tensor modes and contains a solenoidal vector $\omega_i^{(2)}$.

The second-order expression for the temperature fluctuation field in the Poisson gauge has been obtained in Ref. [37], by implementing the general formalism introduced in Ref. [38]. We are interested here in the large-scale limit of that expression, which allows to unambiguously define the primordial non-Gaussian contribution. Keeping only the large-scale limit of the linear and second-order terms in Eqs.(2.27) and (2.28) of Ref. [37], we obtain

$$\frac{\Delta T}{T} = \phi_{\mathcal{E}}^{(1)} + \tau_{\mathcal{E}}^{(1)} + \frac{1}{2}\left(\phi_{\mathcal{E}}^{(2)} + \tau_{\mathcal{E}}^{(2)}\right) - \frac{1}{2}\left(\phi_{\mathcal{E}}^{(1)}\right)^2 + \phi_{\mathcal{E}}^{(1)}\tau_{\mathcal{E}}^{(1)}\,, \tag{79}$$

where $\phi_{\mathcal{E}} = \phi_{\mathcal{E}}^{(1)} + \frac{1}{2}\phi_{\mathcal{E}}^{(2)}$ is the lapse perturbations at emission, $\tau_{\mathcal{E}} = \tau_{\mathcal{E}}^{(1)} + \frac{1}{2}\tau_{\mathcal{E}}^{(2)}$ is the intrinsic fractional temperature fluctuation at emission, $\tau_{\mathcal{E}} \equiv \Delta T/T|_{\mathcal{E}}$. Let us recall that, at linear order $\phi^{(1)} = \psi^{(1)}$. In Eq. (79) we dropped all those terms which represent *integrated* contributions such as Integrated Sachs-Wolfe, Rees-Sciama and more complicated second-order integrated effects [39]. Indeed, the non-linear parameter $f_{\rm NL}$ as introduced in [27,40] singles out the large-scale part of the second-order CMB anisotropies. One should be able to distinguish the secondary integrated terms from the large-scale effects thanks to their specific angular scale dependence. For the very same reason, we disregarded gravitational-lensing, time-delay, Doppler terms and all those second-order effects which are characterized by a high-ℓ harmonic content. We finally dropped contributions at the observer position, which only modify the *monopole* term.

To obtain the intrinsic anisotropy in the photon temperature, we can expand the photon energy-density $\rho_\gamma \propto T^4$ up to second-order and write

$$\tau_{\mathcal{E}}^{(1)} = \frac{1}{4}\frac{\delta^{(1)}\rho_\gamma}{\rho_\gamma}\bigg|_{\mathcal{E}} = \frac{1}{3}\frac{\delta^{(1)}\rho_m}{\rho_m}\bigg|_{\mathcal{E}}\,, \tag{80}$$

where ρ_γ is the mean photon energy-density, and

$$\tau_{\mathcal{E}}^{(2)} = \frac{1}{4}\frac{\delta^{(2)}\rho_\gamma}{\rho_\gamma}\bigg|_{\mathcal{E}} - \frac{3}{16}\left(\frac{\delta^{(1)}\rho_\gamma}{\rho_\gamma}\bigg|_{\mathcal{E}}\right)^2\,. \tag{81}$$

Next, we need to relate the photon energy-density fluctuation to the lapse perturbation, which we can easily do by extending the standard adiabaticity condition to second-order. At first-order the adiabaticity condition reads $\zeta_m^{(1)} = \zeta_\gamma^{(1)}$ and we obtain

$$\frac{\delta^{(1)}\rho_\gamma}{\rho_\gamma} = \frac{4}{3}\frac{\delta^{(1)}\rho_m}{\rho_m}\,, \tag{82}$$

where ρ_m is the average energy-density of the matter component. At second-order the adiabaticity condition imposes $\zeta_m^{(2)} = \zeta_\gamma^{(2)}$. From Eq. (36) applied to matter and radiation we find

$$\frac{\delta^{(2)}\rho_\gamma}{\rho_\gamma} = \frac{4}{3}\frac{\delta^{(2)}\rho_m}{\rho_m} + \frac{4}{9}\left(\frac{\delta^{(1)}\rho_m}{\rho_m}\right)^2\,. \tag{83}$$

In the large-scale limit, the energy constraint, Eqs. (3.3) and (4.4) of Ref. [34] in the matter dominated era, yields

$$\frac{\delta^{(1)}\rho_m}{\rho_m} = -2\psi^{(1)} \tag{84}$$

and

$$\frac{\delta^{(2)}\rho_m}{\rho_m} = -2\phi^{(2)} + 8\left(\psi^{(1)}\right)^2 \,. \tag{85}$$

We finally obtain the fundamental relation[‡]

$$\frac{\Delta T}{T} = \frac{1}{3}\left[\psi_{\mathcal{E}}^{(1)} + \frac{1}{2}\left(\phi_{\mathcal{E}}^{(2)} - \frac{5}{3}\left(\psi_{\mathcal{E}}^{(1)}\right)^2\right)\right] \,. \tag{86}$$

From Eq. (86), it is clear that the expression for the second-order temperature fluctuations is *not* a simple extension of the first-order Sachs-Wolfe effect $\frac{\Delta T^{(1)}}{T} = \frac{1}{3}\psi_{\mathcal{E}}^{(1)}$ to second-order since it receives a correction provided by the term $-\frac{5}{3}\left(\psi_{\mathcal{E}}^{(1)}\right)^2$.

If we express the lapse function at second-order as a general convolution (see, e.g., Ref. [32])

$$\phi = \phi^{(1)} + \frac{1}{2}\phi^{(2)} = \psi^{(1)} + f_{\rm NL}^{\phi} * \left(\psi^{(1)}\right)^2 \,, \tag{87}$$

up to a constant offset, from Eq. (86) we can define the *true* non-linearity parameter $f_{\rm NL}$ which is the quantity actually measurable by high-resolution CMB experiments, after properly subtracting instrumental noise, foreground contributions and small-scale second-order terms. Therefore, we find

$$f_{\rm NL} = f_{\rm NL}^{\phi} - \frac{5}{6} \,. \tag{88}$$

We warn the reader that this is the quantity which enters in the determination of higher-order statistics (as the bispectrum of the temperature anisotropies) and to which the phenomenological study performed in Ref. [27] applies. A number of present and future CMB experiments, such as *WMAP* [26] and *Planck*, have enough resolution to either constrain or detect non-Gaussianity of CMB anisotropy data parametrized by $f_{\rm NL}$ with high precision [27].

‡One can easily obtain a gauge-invariant definition of the temperature fluctuations $\frac{\Delta T}{T}$ in terms of the second-order gauge-invariant lapse function (on large scales)

$$\begin{aligned}\phi^{(2)}_{\rm GI} &= \phi^{(2)} + \omega^{(1)}\left[2\left(\psi^{(1)'} + 2\frac{a'}{a}\psi^{(1)}\right) + \omega^{(1)''}\right.\\ &\left. + \, 5\frac{a'}{a}\omega^{(1)'} + \left(\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right)\omega^{(1)}\right]\\ &+ 2\omega^{(1)'}\left(2\psi^{(1)} + \omega^{(1)'}\right) + \alpha^{(2)'} + \frac{a'}{a}\alpha^{(2)},\end{aligned}$$

where

$$\begin{aligned}\alpha^{(2)} &= \omega^{(2)} + 3\omega^{(1)}\omega^{(1)'}\\ &+ \nabla^{-2}\partial^i\left[-4\psi^{(1)}\partial_i\omega^{(1)} - \omega^{(1)'}\partial_i\omega^{(1)} - 2\omega^{(1)}\partial_i\omega^{(1)'}\right].\end{aligned}$$

Notice that in the Poisson gauge $\phi^{(2)}_{\rm GI} = \phi^{(2)}$.

VII. NON-GAUSSIANITY IN THE VARIOUS SCENARIOS

In each of the scenarios described in the previous section it is possible to calculate the level of non-linearity in the gravitational potential $\phi = \phi^{(1)} + \frac{1}{2}\phi^{(2)}$ (or ψ) in the longitudinal (Poisson) gauge [36]. For instance the gravitational potential ϕ can be expressed in momentum space as (up to a momentum-dependent function which has to be added to account for the condition $\langle\phi\rangle = 0$)

$$\begin{aligned}\phi(\mathbf{k}) &= \phi^{(1)}(\mathbf{k}) + \frac{1}{(2\pi)^3}\int d^3k_1\, d^3k_2\, \delta^{(3)}\left(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}\right)\\ &\times f_{\rm NL}^{\phi}\left(\mathbf{k}_1, \mathbf{k}_2\right)\phi^{(1)}(\mathbf{k}_1)\phi^{(1)}(\mathbf{k}_2)\,,\end{aligned} \tag{89}$$

where we have defined an effective "momentum-dependent" non-linearity parameter $f_{\rm NL}^{\phi}$. Here the linear lapse function $\phi^{(1)} = \psi^{(1)}$ is a Gaussian random field. The gravitational potential bispectrum reads

$$\begin{aligned}\langle\phi(\mathbf{k}_1)\phi(\mathbf{k}_2)\phi(\mathbf{k}_3)\rangle &= (2\pi)^3\,\delta^{(3)}\left(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3\right)\\ &\times\left[2\, f_{\rm NL}^{\phi}\left(\mathbf{k}_1, \mathbf{k}_2\right)\,\mathcal{P}_\phi(k_1)\mathcal{P}_\phi(k_2) + \text{cyclic}\right]\,,\end{aligned} \tag{90}$$

where $\mathcal{P}_\phi(k)$ is the power-spectrum of the gravitational potential. We stress again that $f_{\rm NL}^{\phi}$ defines the non-Gaussianity of the gravitational potential, and it does not define the non-Gaussianity level of the CMB temperature fluctuations, see Eq. (86).

We now give the predictions for the non-Gaussianity in the three scenario considered.

A. Standard scenario

In the one-single field model of inflation the initially tiny non-linearity in the cosmological perturbations generated during the inflationary epoch [32,41] gets enhanced in the post-inflationary stages giving rise to a well-defined prediction for the non-linearity in the gravitational potentials. In Ref. [34] it has been shown how to calculate the second-order gravitational potential at second-order $\phi^{(2)}$ from an inflationary epoch to the radiation and matter dominated epochs by exploiting the conservation on large scales of the curvature perturbation $\zeta^{(2)}$ during the inflationary stage and the radiation/matter phases.

As we have proved in Section V.A, the curvature perturbation $\zeta^{(2)}$ is conserved on large scales even during the reheating phase after inflation. Using the explicit expression for $\zeta^{(2)}$, the second-order energy constraint (Eqs. (4.7) and (4.2) of Ref. [34]) and the traceless part of the (i-j)-components of Einstein's equations at second order (Eq. (43) of Ref. [35]) we obtain, for perturbations re-entering the horizon during the matter-dominated era[§],

§Note that this result coincides with the one given in Ref.

$$f^{\phi}_{\rm NL}(\mathbf{k}_1,\mathbf{k}_2) \simeq -\frac{1}{2} + g(\mathbf{k}_1,\mathbf{k}_2)\ , \tag{91}$$

where

$$g(\mathbf{k}_1,\mathbf{k}_2) = \frac{\mathbf{k}_1\cdot\mathbf{k}_2}{k^2}\left(1+3\frac{\mathbf{k}_1\cdot\mathbf{k}_2}{k^2}\right)\ , \tag{92}$$

where $\mathbf{k}=\mathbf{k}_1+\mathbf{k}_2$. Notice that in the final bispectrum expression, the diverging terms arising from the infrared behaviour of $f^{\phi}_{\rm NL}(\mathbf{k}_1,\mathbf{k}_2)$ are automatically regularized once the monopole term is subtracted from the definition of ϕ (by requiring that $\langle\phi\rangle$=0). Using Eq. (88), we conclude that in the standard scenario where cosmological perturbations are generated by the inflaton field, the value of non-Gaussianity is provided by

$$f_{\rm NL}(\mathbf{k}_1,\mathbf{k}_2) \simeq -\frac{4}{3} + g(\mathbf{k}_1,\mathbf{k}_2)\,. \tag{93}$$

B. Curvaton scenario

In Ref. [35] the level of non-Gaussianity in the gravitational potential ϕ has been calculated following in the sudden decay approximation the evolution of the gauge-invariant curvature perturbation $\zeta^{(2)}$ produced by the initial isocurvature perturbations in the curvaton field σ. In the curvaton scenario we find

$$f^{\phi}_{\rm NL} = \left[\frac{7}{6}+\frac{5}{6}r-\frac{5}{4r}\right] + g(\mathbf{k}_1,\mathbf{k}_2)\,. \tag{94}$$

Using expression (88), we find that the level of non-Gaussianity in the curvaton scenario is provided by

$$f_{\rm NL} = \left[\frac{1}{3}+\frac{5}{6}r-\frac{5}{4r}\right] + g(\mathbf{k}_1,\mathbf{k}_2)\,. \tag{95}$$

C. Inhomogeneous reheating scenario

Using the technique developed in Refs. [34,35] we can now calculate the non linear parameter $f^{\phi}_{\rm NL}$ in the inhomogeneous reheating scenario. We can switch from the spatially flat gauge to the Poisson gauge, since the results obtained in the previous section are gauge-invariant, involving the curvature perturbations. This is evident, for example, from Eq. (78).

During the matter dominated era from Eq. (34) it turns out that [34]

$$\begin{aligned}\zeta^{(2)} &= -\psi^{(2)} + \frac{1}{3}\frac{\delta^{(2)}\rho}{\rho} + \frac{5}{9}\left(\frac{\delta^{(1)}\rho}{\rho}\right)^2 \\ &= -\psi^{(2)} + \frac{1}{3}\frac{\delta^{(2)}\rho}{\rho} + \frac{20}{9}\left(\psi^{(1)}\right)^2\ ,\end{aligned} \tag{96}$$

where in the last step we have used that on large scales $\delta^{(1)}\rho/\rho = -2\psi^{(1)}$ in the Poisson gauge [34]. Eq. (96) combined with Eq. (78) which gives the constant value on large scales of the curvature perturbation $\zeta^{(2)}$ during the matter dominated era, yields

$$\psi^{(2)} - \frac{1}{3}\frac{\delta^{(2)}\rho}{\rho} = \frac{5}{6}\left(\psi^{(1)}\right)^2\ , \tag{97}$$

where we have used the usual relation between the curvature perturbation and the Bardeen potential $\zeta^{(1)} = -\frac{5}{3}\psi^{(1)}$ during the matter dominated era. From the (0-0) and (i-j)-components of Einstein equations, see Eqs. (A.39) and (A.42-43) in Ref. [32], the following relations hold on large scales during the matter dominated epoch

$$\begin{aligned}\phi^{(2)} &= -\frac{1}{2}\frac{\delta^{(2)}\rho}{\rho} + 4\left(\psi^{(1)}\right)^2\ , \\ \psi^{(2)} - \phi^{(2)} &= -\frac{2}{3}\left(\psi^{(1)}\right)^2 + \frac{10}{3}\nabla^{-2}\left(\psi^{(1)}\nabla^2\psi^{(1)}\right) \\ &\quad - 10\,\nabla^{-4}\left(\partial^i\partial_j\left(\psi^{(1)}\partial_i\partial^j\psi^{(1)}\right)\right)\ .\end{aligned} \tag{98}$$

We then read the non-linearity parameter for the gravitational potential $\phi = \phi^{(1)} + \frac{1}{2}\phi^{(2)}$

$$f^{\phi}_{\rm NL} = \frac{3}{4} + g(\mathbf{k}_1,\mathbf{k}_2)\,. \tag{99}$$

Using expression (88), we find that the level of non-Gaussianity in the inhomogeneous reheating scenario where $\Gamma\propto\chi^2$ is provided by

$$f_{\rm NL} = -\frac{1}{12} + g(\mathbf{k}_1,\mathbf{k}_2)\,. \tag{100}$$

D. Some comments

At this point, we would like to make some comments on the determination of the non-linear parameter performed for the curvaton scenario in Ref. [16] and for the inhomogeneous scenario in Refs. [17,18]. Our results differ from those previously obtained for several reasons. First, the value of $f_{\rm NL}$ has been obtained in Refs. [16–18] by extending at second-order the first-order relation $\phi^{(1)} = -\frac{3}{5}\zeta^{(1)}$. This procedure is not correct since this linear relation is lost at second-order. Secondly, the value of $\zeta^{(2)}$ has been obtained for the curvaton scenario in Ref. [16] by merely expanding the energy-density fluctuation ρ at second-order and in the inhomogeneous scenario [17,18] by expanding the decay rate Γ at second-order. However,

[34] up to an integrated over time gradient term which, however, is fully negligible when evaluating the bispectrum of the gravitational potential on large scales.

the gauge-invariant curvature perturbation contains additional second-order terms of the form $(\text{first} - \text{order})^2$, see Eq. (34). Finally, the value of the non-linear parameter $f_{\rm NL}$ has to be defined through the exact relation between the temperature anisotropies and the (gauge-invariant) gravitational potentials, see Eq. (86).

VIII. CONCLUSIONS

In this paper we have presented a general gauge-invariant formalism to study the evolution of curvature perturbations at second-order. In particular, we have addressed the evolution of the total curvature perturbation in the three scenarios for the generation of the cosmological perturbations on large scales: the standard scenario where perturbations are induced by the inflaton field and the curvaton and the inhomogeneous scenarios where the curvature perturbation is produced by initial isocurvature perturbations. We have calculated the exact formula for the second-order temperature fluctuations on large scales extending the well-known expression for the Sachs-Wolfe effect at first-order. We have provided the proper definition of the non-linear parameter entering in the determination of the bispectrum of temperature anisotropies, thus clarifying what is the quantity to be used in the analysis aimed to look for non-Gaussian properties in the CMB anisotropies such as the one given in Ref. [27]. Finally, we have computed the values of non-Gaussianity in the various scenarios for the generation of the cosmological perturbations. According to Ref. [27], the minimum value of $|f_{\rm NL}|$ that will become detectable from the analysis of *WMAP* and *Planck* data, after properly subtracting detector noise and foreground contamination, is about 20 and 5, respectively, and 3 for an ideal experiment [27]. Our findings imply that a future determination of non-Gaussianity at a level much larger than $f_{\rm NL} \sim 5$ would favour the curvaton scenario, while the value of non-Gaussianity provided by the standard scenario is below the minimum value ~ 5 advocated in Ref. [27]. However, such a lower bound has been obtained by analysing the capability of the bispectrum statistics to detect non-Gaussianity of the type discussed in this paper. Alternative statistical estimators based on the multivariate distribution function of the spherical harmonics of CMB maps [42], or wavelet-based analyses [43], together with a systematic application of MonteCarlo-simulated non-Gaussian CMB maps [44], may allow to achieve sensitivity to lower values of $f_{\rm NL}$.

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